

DEFORMING COMPLETE HERMITIAN METRICS WITH UNBOUNDED CURVATURE

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ABSTRACT. We produce solutions to the Kähler-Ricci flow emerging from complete initial metrics g_0 which are C^0 Hermitian limits of Kähler metrics. Of particular interest is when g_0 is Kähler with unbounded curvature. We provide such solutions for a wide class of $U(n)$ -invariant Kähler metrics g_0 on \mathbb{C}^n , many of which having unbounded curvature. As a special case we have the following Corollary: The Kähler-Ricci flow (1.2) has a smooth short time solution starting from any smooth complete $U(n)$ -invariant Kähler metric on \mathbb{C}^n with either non-negative or non-positive holomorphic bisectional curvature, and the solution exists for all time in the case of non-positive curvature.

Keywords: Kähler-Ricci flow, parabolic Monge-Ampère equation, $U(n)$ invariant Kähler metrics

1. INTRODUCTION

Let (M^n, g_0) be a complete noncompact Riemannian manifold. The Ricci flow is the following evolution equation

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} \\ g(0) = g_0. \end{cases}$$

Shi proved in [S1] that suppose the curvature of g_0 is bounded then (1.1) has a solution $g(t)$ up to some time $T > 0$ depending only on the curvature bound for g_0 and the complex dimension n of M such that the curvature is bounded in space-time. If in addition that (M^n, g_0) is a Kähler manifold with complex dimension n , then Shi [S2] proved that the solution $g(t)$ is also Kähler. Hence $g(t)$ satisfies Kähler-Ricci flow equation:

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$$(1.2) \quad \begin{cases} \frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \\ g(0) = g_0. \end{cases}$$

See Theorem 2.1 for more details.

In this work we want to discuss the short time existence and long time existence of the Kähler-Ricci flow (1.2) without the assumption that g_0 has bounded curvature.

There are many results of existence without assuming that the initial data g_0 has bounded curvature. In [Si], Simon proved that starting from any sufficiently small C^0 perturbation g_0 of a complete Riemannian metric with bounded curvature, there is a short time solution of the Ricci flow (1.1). We also refer to the works [KL, SSS1] where the Ricci flow is solved starting with rough initial data obtained from a sufficiently small perturbation of the Euclidean metric on \mathbb{R}^n , and [SSS2] for a similar result for the hyperbolic metrics. In [CW] Cabezas-Rivas and Wilking obtained a short time existence result of the Ricci flow starting from any complete Riemannian metric with nonnegative complex sectional curvature. They do not assume the curvature is bounded and do not assume the initial metric is a small perturbation of a complete metric with bounded curvature. The solutions from [Si], [KL], [SSS1] and [SSS2] are complete and have bounded curvature when $t > 0$. In [CW], complete solutions are constructed where the curvature is bounded whenever $t > 0$ and examples are also given of complete solutions where the curvature is unbounded when $t > 0$.

For Kähler-Ricci flow, when $n = 1$, Geisen-Topping [GT] proved that (1.2) always has a solution starting from any smooth Kähler metric g_0 which may have unbounded curvature, and may even be incomplete. In fact, they also constructed solutions where $g(t)$ is complete with unbounded curvature for all $t \in [0, T)$. Using the construction of [CW], Yang-Zheng proved that if g_0 is a $U(n)$ invariant complete Kähler metric with nonnegative sectional curvature, and with some technical assumptions on the solution $g(t)$ of (1.1), then $g(t)$ is Kähler for $t > 0$. Hence in this case Kähler-Ricci flow (1.2) has short time solution.

In this work, we obtain some existence results of Kähler-Ricci flow (1.2) without assuming g_0 has bounded curvature.

Theorem 1.1. *Let $\{h_{k,0}\}$ be a sequence of smooth Kähler metrics on a complex manifold M^n converging uniformly on compact subsets to a continuous Hermitian metric g_0 . Let \hat{g} be a complete Kähler metric with bounded curvature and suppose $C^{-1}\hat{g} \leq h_{k,0} \leq C\hat{g}$ for some C independent of k and h_k has bounded curvature for every k . Then the*

Kähler-Ricci flow (1.2) has a smooth solution $g(t)$ on $M \times (0, T)$ for some $T > 0$, which is continuous on $M \times [0, T)$ with $g(0) = g_0$, and $g(t)$ has bounded curvature for $t > 0$. If g_0 is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M , then $g(t)$ extends to a smooth solution on $M \times [0, T)$ with $g(0) = g_0$.

With some additional assumptions on $h_{k,0}$, the theorem is still true if we only assume $h_{k,0} \geq C^{-1}\hat{g}$. See Theorems 4.1, 4.2 for more details. We also obtain estimates for the existence time T .

As a corollary, one can prove that the Kähler-Ricci flow (1.2) has short time solution if g_0 is perturbation of a complete Kähler metric \hat{g} with bounded curvature by a potential satisfying certain growth conditions. More precisely, we prove that

Suppose (M^n, \hat{g}) is a complete noncompact Kähler manifold with bounded curvature. Suppose u is a C^2 function and $|\nabla_0 u|_{g_0}$ and $|u|$ are of sublinear growth such that $\hat{g} + \sqrt{-1}\partial\bar{\partial}u$ is uniformly equivalent to \hat{g} . Then the Kähler-Ricci flow (1.2) has a smooth short time solution $g(t)$ on $M \times (0, T)$ for some $T > 0$ such that $g(t)$ is continuous on $M \times [0, t)$ and $g(0) = \hat{g} + \sqrt{-1}\partial\bar{\partial}u$.

Applying the general existence theorems to $U(n)$ invariant Kähler metrics on \mathbb{C}^n , we obtain:

Theorem 1.2. *The Kähler-Ricci flow (1.2) has a smooth short time solution starting from any smooth complete $U(n)$ -invariant Kähler metric g_0 on \mathbb{C}^n with either non-negative or non-positive holomorphic bisectional curvature, and the solution exists for all time in the case of non-positive holomorphic bisectional curvature.*

This gives an affirmative answer to a question posed by Yang-Zheng [YZ]. In fact, one can prove results more general than the Theorem above. See Theorems 5.3, 5.4 and their corollaries for more details. We also obtain some long time existence results for g_0 with nonnegative holomorphic bisectional curvature, see Theorem 5.5

Another consequence of our general existence results is the following estimate for the existence time for (1.2) on (M, g_0) :

Suppose that $g_1 \leq g_0$ where g_1 has bounded holomorphic bisectional curvature, bounded above by K and that in addition either $g_0 \leq Cg_1$ for some $C > 0$ or $g_0 \leq g_2$ where the holomorphic bisectional curvature of g_2 is non-positive. Then the Kähler-Ricci flow has a solution $g(t)$ with initial data g on $M \times [0, T = 1/2nK)$, where by convention we take $T = \infty$ when $K = 0$.

See Corollary 4.1 and 4.3 for details. Note that the estimate does not depend on the curvature of g_0 .

The organization of the paper is as follow. In section §2 we review some basic theory and estimates for (1.2), and in §3 we prove some further a priori estimates which we will need. §4 contains our main existence theorems Theorems 4.2 and 4.1 and accompanying corollaries. In §5 we review Wu-Zheng's description in [WZ] of $U(n)$ invariant Kähler metrics on \mathbb{C}^n and use this to apply our previous results to prove Theorems 5.3 and 5.4.

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2. PRELIMINARIES

In this section, we review some well known results for the Kähler Ricci flow which we will use in the paper. We first recall Shi's short time existence theorem for (1.2) in [S2].

Theorem 2.1. *Suppose \tilde{g} is a complete Kähler metric on a noncompact complex manifold with curvature bounded by a constant K . Then there exists $0 < T \leq \infty$ depending only on K and the dimension n , and a smooth solution $g(t)$ to (1.2) on $M \times [0, T)$ with initial condition $g(0) = \tilde{g}$ such that*

- (i) $g(t)$ is Kähler and equivalent to \tilde{g} for all $t \in [0, T)$;
- (ii) $g(t)$ has uniformly bounded curvature on $M \times [0, T']$ for all $0 < T' < T$. In particular, for any $l \geq 0$ there exists a constant C_l depending only on l , \tilde{g} and the dimension n such that

$$\sup_M |\nabla^l \text{Rm}(h(t))|_{h(t)}^2 \leq \frac{C_l}{t^l},$$

on $M \times [0, T']$.

- (iv) If $T < \infty$ and $\limsup_{t \rightarrow T} \sup_M |\text{Rm}(x, t)| < \infty$, then the $g(t)$ can be extended, as a solution to (1.2), beyond T to $T_1 > T$ so that (ii) is still true with T replaced by T_1 .

The solution $g(t)$ in Theorem 2.1 has bounded curvature on $M \times [0, T)$. From this it is easy to see that $g(0)$ and $g(t)$ are uniformly equivalent. On the other hand, it is a well known fact one can study the Kähler-Ricci flow (1.2) through the parabolic Monge-Ampère equation. This was originated in [C], and we refer to [CT] and references therein for further details on this fact. By the Evans-Krylov theory [Kr, Ev] for fully non-linear equations, which takes the form in the following Theorem 2.2 in case of (1.2) (see [SW] for a proof using only

the maximum principle), one may conclude that if $g(0)$ and $g(t)$ are uniformly equivalent, one can obtain curvature bounds. Actually, we need a more general version in the sense that we need local estimates and $g(t)$ is assumed to be uniformly equivalent to a fixed background metric \hat{g} .

Let us first fix some notations and terminology. (M^n, \hat{g}) is said to have bounded geometry of infinite order if the curvature tensor and all its covariant derivatives are uniformly bounded. Also, we will denote the geodesic ball with respect to the metric g with center at p and radius r by $B_g(p, r)$. The following theorem can be found in [SW].

Theorem 2.2. *Let (M^n, \hat{g}) be a complete noncompact Kähler manifold with bounded geometry of infinite order. Let $h(t)$ be a solution of Kähler-Ricci (1.2) on $M \times [0, T)$ with initial data h_0 which is a complete Kähler metric. For any $x \in M$, suppose there is a constant $N > 0$, such that*

$$N^{-1}\hat{g} \leq h(t) \leq N\hat{g}$$

on $B_{\hat{g}}(x, 1) \times [0, T)$. Then

(i)

$$|\hat{\nabla}^k h|_{\hat{g}}^2 \leq \frac{C_k}{t^k}$$

on $B_{\hat{g}}(x, 1/2) \times (0, T)$, for some constant C_k depending only on k, \hat{g}, n, T and N .

(ii) If we assume $|\hat{\nabla}^k h_0|_{\hat{g}}^2$ is bounded in $B_{\hat{g}}(x, 1)$ by c_k , for $k \geq 1$, then

$$|\hat{\nabla}^k h|_{\hat{g}}^2 \leq C_k,$$

on $B_{\hat{g}}(x, 1/2) \times [0, T)$ for some constant C_k depending only on k, c_k, n, T and N .

Proof. Since \hat{g} has bounded geometry of infinite order, by [TY], for any $x \in M$ there exists a local biholomorphism $\phi_x : D \rightarrow M$, where $D = D(1)$ is the open unit ball in \mathbb{C}^n , satisfying the following in D

- (a) $\phi_x(0) = x$, $\phi_x(D) \subset \hat{B}(x, 1)$, $\phi_x(D) \supset \hat{B}(x, 2\delta)$ for some $\delta > 0$ which is independent of x .
- (b) $C^{-1}\delta_{i\bar{j}} \leq (\phi_x^*(\hat{g}))_{i\bar{j}} \leq C\delta_{i\bar{j}}$ for some C independent of x .
- (c) $\left| \frac{\partial^l (\phi_x^*(\hat{g}))_{i\bar{j}}}{\partial z^L} \right| \leq C_l$ for any l, i, j and multi index L of length l for some constant C_l which is independent of x .

Consider $\phi_x^*(h(t))$, which clearly will solve 1.2 on $D(1) \times [0, T)$. By the Evans-Krylov theory [Ev], [Kr] for fully non-linear elliptic and parabolic equations (see also [SW] for a maximum principle proof in the case of Kähler Ricci flow), the result follows. \square

We end this section with the following longtime existence Theorem from [CT] when we look at certain special solutions to (1.2) on \mathbb{C}^n in Theorem 5.5.

Theorem 2.3. *Let (M, g_0) be a complete non-compact Kähler manifold such that*

- i) $|\text{Rm}_0(x)| \rightarrow 0$ as $d(x) \rightarrow \infty$ where $d(x)$ is the distance function on M from some $p \in M$.
- ii) *The injectivity radius of (M, g_0) is uniformly bounded below by some constant $c > 0$.*
- iii) *There exists a strictly pluri-subharmonic function F on M .*

Then the Kähler-Ricci flow $g' = -\text{Ric}$ has a longtime solution $g(t)$ on M with initial condition $g(0) = g_0$. Moreover, the curvature of $g(t)$ is bounded uniformly on $M \times [0, T]$ for all $T < \infty$.

3. FURTHER ESTIMATES

In this section we prove some further estimates which we will require to prove our main theorems. One main tool is Theorem 2.2. Hence we want to obtain C^0 estimate of solution of Kähler-Ricci flow in terms of a back ground metric.

First recall that the holomorphic bisectional curvature of a Kähler manifold is bounded above by K if

$$(3.1) \quad \frac{R(X, \bar{X}, Y, \bar{Y})}{\|X\|^2 \|Y\|^2 + |\langle X, \bar{Y} \rangle|^2} \leq K$$

for any two nonzero $(1,0)$ -vectors X, Y . The holomorphic bisectional curvature of a Kähler manifold is bounded below by K is defined similarly.

Lemma 3.1. *Let \hat{g} be a complete Kähler metric on M with bounded curvature, and holomorphic bisectional curvature bounded above by K . Let $h(t)$ be a solution to (1.2) on $M^n \times [0, T_0)$ with $h(0) = h_0$ such that $h(t)$ has uniformly bounded curvature for on $M \times [0, T']$ for all $0 < T' < T_0$. Let $T = \frac{1}{2nK}$ where $T = \infty$ if $K = 0$ by convention.*

- (i) *Suppose $h_0 \geq \hat{g}$. Then $h(t) \geq v(t)\hat{g}$ on $M \times [0, \min\{T_0, T\})$, where $v(t) = \frac{1}{n} - 2Kt$.*
- (ii) *Suppose in addition to (i) we have $h_0 \leq C\hat{g}$, that is if $\hat{g} \leq h_0 \leq C\hat{g}$, then*

$$(1 - w(t))\hat{g} \leq h(t) \leq (1 + w(t))\hat{g}$$

on $M \times [0, \min\{T_0, T\})$,

where $w(t) = \sqrt{v_2(t)(v_1(t) + v_2(t) - 2n)}$, and

$$v_1(t) = \frac{1}{v(t)} = \frac{1}{\frac{1}{n} - 2Kt}, v_2(t) = nCe^{2Kv_1(t)t}$$

In particular, we have $\lim_{t \rightarrow 0} w(t) = n\sqrt{C(C-1)}$.

Proof. (i) Let $\phi(t) := \text{tr}_{h(t)} \hat{g}$. Let $\square = \frac{\partial}{\partial t} - \Delta$, where Δ is the Laplacian with respect to $h(t)$. Then as in [ST], we can calculate in a normal coordinate relative to $h(t)$ and use (1.2) to get

$$\begin{aligned} \square \phi &= ((h_t)^{i\bar{j}} \hat{g}_{i\bar{j}}) - h^{k\bar{l}} (h^{i\bar{j}} \hat{g}_{i\bar{j}})_{k\bar{l}} \\ (3.2) \quad &= (R^{i\bar{j}} \hat{g}_{i\bar{j}}) - (R^{i\bar{j}} \hat{g}_{i\bar{j}}) + h^{k\bar{l}} h^{i\bar{j}} \hat{R}_{i\bar{j}k\bar{l}} - \hat{g}^{p\bar{q}} h^{k\bar{l}} h^{i\bar{j}} \partial_k \hat{g}_{i\bar{q}} \partial_{\bar{l}} \hat{g}_{p\bar{j}} \\ &\leq 2K\phi^2. \end{aligned}$$

Now $v_1(t) := \frac{1}{v(t)}$ is the positive solution to the ODE

$$\frac{dv_1(t)}{dt} = 2Kv_1^2(t); \quad v_1(0) = n$$

for $t \in [0, T]$. Let $S \in (0, \min\{T_0, T\})$ be fixed. Since $h(t)$ has uniformly bounded curvature on $M \times [0, S]$ we have $h(t) \geq C_1 h_0$ for some $C_1 > 0$ and hence ϕ is a bounded function on $M \times [0, S]$. Moreover, v_1 is also a bounded function on $M \times [0, S]$. Let $A = \sup_{M \times [0, S]} (\phi + v_1)$. Then on $M \times [0, S]$

$$\begin{aligned} \square (e^{-(2AK+1)t} (\phi - v_1)) &\leq e^{-(2AK+1)t} [2K(\phi^2 - v_1^2) - (2AK+1)(\phi - v_1)] \\ &= e^{-(2AK+1)t} [2K(\phi + v_1) - (2AK+1)] (\phi - v_1) \end{aligned}$$

which is nonpositive at the points where $\phi - v_1 \geq 0$. Using the fact that $h(t)$ has uniformly bounded curvature on $M \times [0, S]$ and the fact that $e^{-(2AK+1)t} (\phi - v_1) \leq 0$ at $t = 0$, which is uniformly bounded on $M \times [0, S]$, we conclude that $e^{-(2AK+1)t} (\phi - v_1) \leq 0$ on $M \times [0, S]$ by the maximum principle, see [NT, Theorem 1.2] for example. This proves (i).

(ii) Let $\psi(t) := \text{tr}_{\hat{g}} h(t)$. For any fixed $S \in [0, \min\{T_h, T\})$, as in [C] we calculate in a normal coordinate relative to \hat{g} and use (1.2) to get that on $M \times [0, S]$:

$$\begin{aligned}
(3.3) \quad \square\psi &= (\hat{g}^{i\bar{j}}(h_t)_{i\bar{j}}) - h^{k\bar{l}}(\hat{g}^{i\bar{j}}h_{i\bar{j}})_{k\bar{l}} \\
&= -(\hat{g}^{i\bar{j}}R_{i\bar{j}}) - h^{k\bar{l}}(\hat{R}_{k\bar{l}}^{i\bar{j}}h_{i\bar{j}}) + (\hat{g}^{i\bar{j}}R_{i\bar{j}}) - \hat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}} h_{k\bar{q}} \\
&= -h^{k\bar{l}}h_{i\bar{j}}\hat{R}_{k\bar{l}}^{\bar{j}i} - \hat{g}^{i\bar{j}}h^{p\bar{q}}h^{k\bar{l}}\partial_i h_{p\bar{l}}\partial_{\bar{j}} h_{k\bar{q}} \\
&\leq 2Kv_1(t)\psi \\
&\leq 2Kv_1(S)\psi
\end{aligned}$$

by (i). Let $w_S(t) = nCe^{2Kv_1(S)t}$ be the solution to the ODE

$$\frac{dw_S(t)}{dt} = 2Kv_1(S)w_S(t); \quad w_S(0) = nC.$$

Then arguing as above, we have $\psi \leq w_S$ on $M^n \times [0, S]$. In particular, we get $\psi(S) \leq w_S(S)$ for every $S \in [0, \min\{T_0, T\})$.

So far, we have $\phi(t) \leq v_1(t)$, and $\psi(t) \leq v_2(t)$ on $M \times [0, \min\{T_0, T\})$ where v_1, v_2 are as in the statement of the Lemma. Now we follow an idea from [S1]. At any point in $(p, t) \in M \times [0, \min\{T_0, T\})$, let λ_i 's be the eigenvalues of h with respect to \hat{g} , and calculate at (p, t)

$$\begin{aligned}
(3.4) \quad \sum_{i=1}^n \frac{1}{\lambda_i} (1 - \lambda_i)^2 &= \sum_{i=1}^n \frac{1}{\lambda_i} + \lambda_i - 2 \\
&\leq \phi + \psi - 2n \\
&\leq v_1(t) + v_2(t) - 2n
\end{aligned}$$

and thus for any fixed i we have

$$(3.5) \quad -w(t) \leq \lambda_i - 1 \leq w(t)$$

where $w(t) = \sqrt{v_2(t)(v_1(t) + v_2(t) - 2n)}$. The conclusion in (ii) then follows. \square

The following lemma basically says that if a local solution $h(t)$ to (1.2) is a priori uniformly equivalent to a fixed metric \hat{g} in space time, and close to \hat{g} at time $t = 0$, then it remains close to \hat{g} in a uniform space time region. Note that in contrast to Lemma 3.1, the a priori assumption here is on $h(t)$ for all t .

Lemma 3.2. *Let $h(t)$ be a smooth solution to (1.2) on $B(1) \times [0, T]$ with $h(0) = h_0$ where $B(1)$ is the unit Euclidean ball in \mathbb{C}^n . Let \hat{g} be a smooth Kähler metric on $B(1)$. Suppose*

$$(3.6) \quad N^{-1}\hat{g} \leq h(t) \leq N\hat{g}$$

on $B(1) \times [0, T)$ for some $N > 0$, and that

$$(3.7) \quad \hat{g} \leq h_0 \leq C\hat{g}$$

on $B(1)$. Then there exists a positive continuous function $a(t) : [0, T) \rightarrow \mathbb{R}$ depending only on \hat{g}, N, C and n such that

$$(3.8) \quad \frac{(1 - a(t))}{C} h_0 \leq h \leq (1 + a(t)) h_0$$

on $B(1/2) \times [0, T)$, where $\lim_{t \rightarrow 0} a(t) = n\sqrt{C(C-1)}$.

Proof. As in the previous Lemma, let $\phi = \text{tr}_h \hat{g}$, $\psi = \text{tr}_{\hat{g}} h$ on $B(1) \times [0, T_0)$. Choose some smooth non-negative cutoff function on $\eta : B(1) \rightarrow \mathbb{R}$ satisfying $\eta|_{B(1/2)} = 1$, $\eta|_{(B(1/2))^c} = 0$, $|\hat{\nabla} \eta|^2 \leq C_1 \eta$, $|\hat{\Delta} \eta|^2 \leq C_2$ on $B(1)$ for some constants C_1, C_2 depending only on \hat{g} . Now we consider the function $\eta\phi$ on $B(1) \times [0, T)$. Then $\eta\phi$ is always zero off $B(1/2)$, and in $B(1/2) \times [0, T)$, using the above estimates and the proof of Lemma 3.1 (i) we obtain

$$(3.9) \quad \begin{aligned} (\partial_t - \Delta)(\eta\phi) &= \eta(\partial_t - \Delta_t)\phi - 2 \langle \nabla \eta, \nabla \phi \rangle - \phi \Delta \eta \\ &= \eta(\partial_t - \Delta_t)\phi - 2 \frac{\langle \nabla \eta, \nabla(\eta\phi) \rangle}{\eta} + \frac{|\nabla \eta|^2}{\eta} \phi - \phi \Delta \eta \\ &\leq \eta(C_3 \phi^2 + C_3) - 2 \frac{\langle \nabla \eta, \nabla(\eta\phi) \rangle}{\eta} + 2NC_1 \phi + NC_2 \phi \\ &\leq C_4(N^2 + 1) - 2 \frac{\langle \nabla \eta, \nabla(\eta\phi) \rangle}{\eta} \end{aligned}$$

where the constants C_3, C_4 depend only on \hat{g}, N, C and n . We conclude from the maximum principle that

$$\eta\phi \leq n + C_4(N^2 + 1)t =: \tilde{v}_1(t)$$

on $B(1) \times [0, T)$.

Now consider the function $\eta\psi$ on $B(1) \times [0, T)$. Then $\eta\psi$ is always zero off $B(1/2)$, and in $B(1/2) \times [0, T)$, using the proof of Lemma 3.1 (ii) and estimating as above we obtain

$$(3.10) \quad \eta\psi \leq nC + C_5(N^2 + 1)t =: \tilde{v}_2(t)$$

on $B(1) \times [0, T)$ for some constants C_5 depending only on \hat{g}, N, C and n .

Now at any point in $(p, t) \in B(1/2) \times [0, T)$, let λ_i 's be the eigenvalues of h with respect to \hat{g} . Then as in the proof of Lemma 3.1 (ii) we get that at (p, t)

$$(3.11) \quad -\tilde{w}(t) \leq \lambda_i - 1 \leq \tilde{w}(t)$$

where $\tilde{w}(t) = \sqrt{\tilde{v}_2(t)(\tilde{v}_1(t) + \tilde{v}_2(t) - 2n)}$. The lemma follows easily from this. \square

In contrast to the previous lemma, in the following lemmas we only assume a lower bound on a solution $h(x, t)$ to (1.2).

Lemma 3.3. *Let $h(x, t)$ be a smooth solution to (1.2) on $M \times [0, T)$ with $h(0) = h_0$. Let $p \in M$. Suppose there is a positive continuous function $\alpha(t) : [0, T) \rightarrow \mathbb{R}$ such that*

$$h(t) \geq \alpha(t)\hat{g}.$$

where \hat{g} is complete with bounded curvature. Then, there exists a positive continuous function $\beta(r, t) : [1, \infty) \times [0, T) \rightarrow \mathbb{R}$ $\beta(r, t)$ depending only on \hat{g} the upper bound of $\text{tr}_{\hat{g}}h_0$ in $B_{\hat{g}}(p, 2r)$, the lower bound of scalar curvature $R(0)$ of $h(0)$ in $B_{\hat{g}}(p, 2r)$, $\alpha(t)$ and the dimension n such that for $r \geq 1$

$$h(t) \leq \beta(r, t)\hat{g}.$$

in $B_{\hat{g}}(p, r) \times [0, T)$.

Proof. Let us assume for simplicity that the distance function $d(x)$ on M with respect to $p \in M$ and the metric \hat{g} , is smooth, as otherwise we may find another positive smooth function which is equivalent of $d(x)$ for $d(x) \geq 1$, see [S2, T]. Let $\phi(s)$ be smooth function on \mathbb{R} such that $\phi = 1$ for $s \leq 1$ and is zero for $s \geq 2$. Moreover, we assume $\phi' \leq 0$, $(\phi')^2/\phi \leq C_1$, $|\phi''| \leq C_2$. Let R be the scalar curvature of $h(t)$. Then

$$(3.12) \quad \left(\frac{\partial}{\partial t} - \Delta \right) R \geq \frac{1}{n} R^2.$$

on $M \times [0, T)$. Let $\varphi(x) = \phi(d(x)/r)$. Then $\varphi(x) = 0$ if $d(x) \geq 2r$. Fix some $T' < T$. Then we compute

$$\begin{aligned} |\nabla \varphi|^2 &= \frac{1}{r^2} (\phi')^2 |\nabla d|^2 \\ &= \frac{1}{r^2} (\phi')^2 h^{\alpha\beta} d_{\alpha} d_{\beta} \\ &\leq \frac{1}{r^2} (\phi')^2 \alpha(t)^{-1} \hat{g}^{\alpha\beta} d_{\alpha} d_{\beta} \\ &\leq C_3 \frac{1}{r^2} (\phi')^2 \end{aligned}$$

on $B(2r) \times [0, T']$ for some constant C_3 depending only on $T', \alpha(t)$ and \hat{g} . Similarly,

$$\begin{aligned} |\Delta\varphi| &= \left| \frac{1}{r} \phi' \Delta d + \frac{1}{r^2} \phi'' |\nabla d|^2 \right| \\ &\leq \left| \frac{1}{r} \phi' h^{\alpha\bar{\beta}} d_{\alpha\bar{\beta}} \right| + \frac{1}{r^2} C_4 \end{aligned}$$

on $B(2r) \times [0, T']$ where C_4 depends on $C_1, C_2, T', \alpha(t)$ and \hat{g} . In order to estimate $h^{\alpha\bar{\beta}} d_{\alpha\bar{\beta}}$. Choose normal coordinates at a point with respect to \hat{g} so that $h^{\alpha\bar{\beta}} = \lambda_a^{-1} \delta_{\alpha\bar{\beta}}$, and thus $\lambda_a^{-1} \leq \alpha^{-1}(t)$ at that point for all a . In these coordinates, we also have $|d_{\alpha\bar{\beta}}| \leq |\nabla_{\hat{g}}^2 d| \leq C_5$ for some constant C_5 depending only on \hat{g} since \hat{g} has bounded curvature, and hence

$$(3.13) \quad |\Delta\varphi| \leq C_6 \left(\frac{1}{r} + \frac{1}{r^2} \right)$$

on $B(2r) \times [0, T']$ where C_6 depends only on C_5 and C_4 . Now

$$\begin{aligned} (3.14) \quad \left(\frac{\partial}{\partial t} - \Delta \right) (\varphi R) &= \varphi \left(\frac{\partial}{\partial t} - \Delta \right) R - R \Delta\varphi - 2 \langle \nabla R, \nabla \varphi \rangle \\ &\geq \frac{1}{n} \varphi R^2 - C_7 |R| - 2 \langle \nabla R, \nabla \varphi \rangle \end{aligned}$$

on $B(2r) \times [0, T']$ where C_7 depends only on C_6 and r . Suppose the infimum of φR on $B(2r) \times [0, T']$ is attained at $t = 0$, then $R \geq \min\{0, \inf_{B_{\hat{g}}(p, 2r)} R(h_0)\}$ on $B_{\hat{g}}(r)$. Suppose instead that φR attains a negative minimum at some $(x, t) \in B(2r) \times [0, T']$ where $t > 0$. Then at (x, t) , $\nabla R = -\frac{R \nabla \varphi}{\phi}$. Hence at this point,

$$(3.15) \quad 0 \geq \frac{1}{n} \varphi R^2 - C_8 |R|$$

where C_8 depends only on C_7, C_3 and r . Hence

$$\varphi^2 |R| \leq n C_8.$$

on $B(2r) \times [0, T']$ and we conclude that $R \geq -C_9$ on $B_{\hat{g}}(p, r) \times [0, T']$ for some C_9 depending only on $T', \hat{g}, r, \alpha(t)$. On the other hand,

$$\frac{\partial}{\partial t} \log \left(\frac{\det(h_{\alpha\bar{\beta}})(t)}{\det(h_{\alpha\bar{\beta}}(0))} \right) = -R \leq C_9.$$

So

$$\det(h_{\alpha\bar{\beta}})(t) \leq e^{C_9 t} \frac{\det(h_{\alpha\bar{\beta}})(0)}{\det(\hat{g}_{\alpha\bar{\beta}})} \det(\hat{g}_{\alpha\bar{\beta}}).$$

on $B_{\hat{g}}(p, r) \times [0, T']$. Let λ_i be eigenvalues of $h(t)$ with respect to \hat{g} . By part (i), $\lambda_i(x, T') \geq \alpha(T')$ for each i and $x \in B_{\hat{g}}(p, r)$, and the above inequality then implies $\lambda_i(x, T') \geq \beta(r, T')$ for some $\beta(r, T')$

depending only on the those constants listed in the Lemma. Moreover, it is not hard to see that $\beta(r, T')$ can be chosen to depend continuously on r, T' as $\alpha(t)$ is continuous. The Lemma follows as T' was chosen arbitrarily. \square

Though we will not need it for the purpose of our applications, it is not hard to see that the previous proof can be adapted easily to give the following local version of the previous Lemma

Lemma 3.4. *Let $h(x, t)$ be a smooth solution to (1.2) on $B(1) \times [0, T)$ with $h(0) = h_0$ where $B(1)$ is the open unit ball in \mathbb{C}^n . Suppose there is a positive continuous function $\alpha(t) : [0, T) \rightarrow \mathbb{R}$ such that*

$$h(t) \geq \alpha(t)\hat{g}_e.$$

where g_e is the standard Euclidean metric on $B(1)$. Then there exists a positive function $\beta(t) : (0, \infty) \times [0, T) \rightarrow \mathbb{R}$ such that $\beta(r, t)$ depends only on the upper bound of $\text{tr}_{g_e} h_0$ in $B(1)$, the lower bound of scalar curvature $R(0)$ of $h(0)$ in $B(1)$, $\alpha(t)$ and the dimension n

$$h(t) \leq \beta(t)\hat{g}.$$

in $B(1/2) \times [0, T)$.

4. KÄHLER RICCI FLOW: GENERAL EXISTENCE THEOREMS

We are now ready to state and prove our main existence Theorems for (1.2) using the estimates in the previous section. Theorems 4.1 and 4.2 provide general existence Theorems for (1.2) when the initial Kähler metric is realized as a limit of a sequence of Kähler metrics satisfying certain properties. The curvature of the initial metric may be unbounded or even undefined. As an application, we obtain an existence result for Kähler metric which is some perturbation of a complete Kähler metric with bounded curvature. In Corollaries ?? and 4.3 we apply the above Theorems to provide an estimate for the maximal existence time for (1.2) assuming that the curvature of the initial metric is bounded. In some cases, we have long time existence.

In the following, we say that a sequence of smooth metrics h_k converge smooth to a metric g on a set U , if h_k converge to g in C^∞ norm on U .

Theorem 4.1. *Let $\{h_{k,0}\}$ be a sequence of smooth Kähler metrics on a complex manifold M converging uniformly on compact subsets to a continuous Hermitian metric g_0 such that:*

- (i) $h_{k,0} \geq \hat{g}$ for all k and some complete Kähler metric \hat{g} with bounded curvature, and holomorphic bisectional curvature bounded above by K ;

- (ii) for every k , Kähler-Ricci flow (1.2) has smooth solution $h_k(t)$ with initial value $h_{k,0}$ on $M \times [0, T']$ for some $T' > 0$ independent of k such that the curvature of $h_k(t)$ is uniformly bounded on $M \times [0, T_1]$ for all $0 < T_1 < T'$;
- (iii) The scalar curvature R_k of $h_{k,0}$ satisfies: for any $r > 0$, there exists a constant $C_r > 0$ such that $R_k \geq -C_r$ on $B_{\hat{g}}(p, r)$ for some fixed point $p \in M$ and all k .

Then the Kähler-Ricci flow (1.2) has a solution $g(t)$ on $M \times [0, T)$ in the sense that $g(t)$ is smooth on $M \times (0, T)$, satisfies the Kähler-Ricci flow equation and $g(t)$ converge to g_0 uniformly on compact sets as $t \rightarrow 0$. Here $T = \min\{T', \frac{1}{2nK}\}$ where we take $T = T'$ if $K = 0$ by convention. Moreover $g(t)$ satisfies $g(t) \geq (1/n - 2nK)\hat{g}$ on $M \times (0, T)$.

In addition, if g_0 is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M , then the solution $g(t)$ extends to a smooth solution on $M \times [0, T)$ with $g(0) = g_0$.

Proof. By Lemma 3.1, we have

$$(4.1) \quad h_k(t) \geq \left(\frac{1}{n} - 2Kt\right) \hat{g}$$

where K as long as $t < T_0 = 1/(2nK)$. By Theorem 2.1, let $\hat{g}(t)$ be the solution Kähler-Ricci flow in the theorem with initial data \hat{g} . Then for any $1 > \epsilon > 0$ small, choose $0 < t_0$ small enough so that $(1 - \epsilon)\hat{g}(t_0) \leq \hat{g} \leq (1 + \epsilon)\hat{g}(t_0)$. Then we have

$$(4.2) \quad h_k(t) \geq \left(\frac{1}{n} - 2Kt\right) (1 - \epsilon)\hat{g}(t_0)$$

and $\hat{g}(t_0)$ has bounded geometry of infinite order. By Lemma 3.3, for there is a positive continuous function $\beta(r, t) : [1, \infty) \times [0, T_0) \rightarrow \mathbb{R}$ such that for $r \geq 1$

$$(4.3) \quad h_k(t) \leq \beta(r, t)\hat{g}(t_0).$$

in $\hat{B}(p, r) \times [0, T)$ where $T = \min\{T', \frac{1}{2nK}\}$ and $p \in M$ is a fixed point. We conclude from Theorem 2.2 (i), that passing to some subsequence, the $h_k(t)$'s converge to a solution $g(t)$ of Kähler-Ricci on $M \times (0, T)$ so that (4.1) is true. Moreover, if g_0 is smooth and $\{h_k\}$ converges smoothly uniformly to g_0 on compact sets, we see from Theorem 2.2 (ii) that in fact $g(t)$ extends to a smooth solution on $M \times [0, T)$ such that $g(0) = g_0$.

We now prove $g(t)$ converge uniformly on compact set to g_0 as $t \rightarrow 0$ when g_0 is only assumed to be continuous. Fix any $x \in M$ and a local biholomorphism $\phi : B(1) \rightarrow M$ where $B(1)$ is the open unit ball in \mathbb{C}^n , and $\phi(0) = x$. Consider the pullbacks $\phi^*h_k(t)$, $\phi^*h_k = \phi^*h_k(0)$,

$\phi^*\hat{g}$, which by abuse of notation we will simply denote by $h_k(t)$, h_k , \hat{g} , respectively, for the remainder of proof. In particular, $h_k(t)$ solves Kähler-Ricci flow (1.2) on $B(1) \times [0, T)$.

Now by our hypothesis on the convergence of h_k , given any $\delta > 0$ we may find k_0 such that $|h_{k_0,0} - g_0|_{\hat{g}} \leq \delta$ and if $k \geq k_0$, then

$$(4.4) \quad (1 - \delta)h_{k_0,0} \leq h_{k,0} \leq (1 + \delta)h_{k_0,0}$$

and by (4.1) and (4.3), there is $N > 0$ such that

$$(4.5) \quad N^{-1}h_{k_0}(t) \leq h_k(t) \leq Nh_{k_0}(t)$$

in $B(t) \times [0, T'/2)$ for all $k \geq k_0$. By Lemma 3.2, there exists a continuous function $a(t)$ depending on N, h_{k_0} and δ such that

$$(1 - a(t))(1 - \delta)h_{k_0}(t) \leq h_k(t) \leq (1 + a(t))(1 - \delta)h_{k_0}(t)$$

in $B(\frac{1}{2}) \times [0, T'/2)$ with $a(0) = n\sqrt{C(C-1)}$, with $C = (1 + \delta)/(1 - \delta)$. Note that $a(t)$ is independent of t . Let $k \rightarrow \infty$, we have

$$(1 - a(t))(1 - \delta)h_{k_0}(t) \leq g(t) \leq (1 + a(t))(1 - \delta)h_{k_0}(t)$$

in $B(\frac{1}{2}) \times (0, T/2)$. Let $t \rightarrow 0$,

$$\begin{aligned} & \limsup_{t \rightarrow 0} |g(t) - g_0|_{\hat{g}} \\ & \leq \limsup_{t \rightarrow 0} (|g(t) - h_{k_0}(t)|_{\hat{g}} + |h_{k_0}(t) - h_{k_0,0}|_{\hat{g}} + |h_{k_0,0} - g_0|_{\hat{g}}) \\ & \leq [(1 + a(0))(1 - \delta) - 1] h_{k_0,0} + \delta \end{aligned}$$

Since $a(0) \rightarrow 0$ as $\delta \rightarrow 0$, we conclude that

$$\limsup_{t \rightarrow 0} |g(t) - g_0|_{\hat{g}} = 0.$$

on $B(\frac{1}{2})$. Hence $g(t)$ converge to g_0 uniformly on compact sets as $t \rightarrow 0$. \square

We do not have any bound on the curvature of solution $g(t)$ in the previous theorem. Also in the previous theorem, we assume that the Kähler-Ricci flow (1.2) has solution with initial data $h_{k,0}$ on a fixed time interval independent of k . We want to remove this assumption and obtain curvature bound for the solutions. In order to do this, we assume $h_{k,0}$ also has an uniform upper bound.

Theorem 4.2. *Let $\{h_{k,0}\}$ be a sequence of smooth Kähler metrics on a complex manifold M^n converging uniformly on compact subsets to a continuous Hermitian metric g_0 such that*

- (i) $C^{-1}\hat{g} \leq h_{k,0} \leq C\hat{g}$ for some C independent of k and some complete Kähler metric \hat{g} with bounded curvature, and holomorphic bisectional curvature bounded above by K .

(ii) h_k has bounded curvature for every k .

Let $T = 1/(2CnK)$ where we take $T = \infty$ if $K = 0$ by convention. Then the Kähler-Ricci flow (1.2) has a smooth solution $g(t)$ on $M \times (0, T)$ such that

- (a) $(1/(nC) - 2Kt)\hat{g} \leq g(t) \leq B(t)\hat{g}$ on $M \times [0, T)$ for some positive continuous function $B(t)$ depending only on C , \hat{g} and n .
- (b) $g(t)$ has bounded curvature for $t > 0$. In particular, for any $0 < T' < T$ and for any $l \geq 0$ there exists a constant C_l depending only on l , T' , \hat{g} and the dimension n such that

$$\sup_M |\nabla^l \text{Rm}(h(t))|_{h(t)}^2 \leq \frac{C_l}{t^{l+2}},$$

- (c) $g(t)$ converges uniformly on compact subsets to g_0 as $t \rightarrow 0$.

Moreover, if g_0 is smooth and $\{h_{k,0}\}$ converges smoothly and uniformly on compact subsets of M , then $g(t)$ extends to a smooth solution on $M \times [0, T)$ with $g(0) = g_0$.

Proof. For each k , let $h_k(t)$ be the solution to (1.2) with initial condition h_k from Theorem 2.1 which is defined on $M \times [0, T_k)$ for some $T_k > 0$. We first claim that there is such that $T_k \geq T$ for all k , where $T = 1/(2nCK)$. By Lemma 3.1, there is a positive continuous function $B(t) : [0, T) \rightarrow \mathbb{R}$ independent of k such that

$$(1/n - 2nCKt)\hat{g} \leq h_k(t) \leq B(t)\hat{g}$$

in $M \times [0, \min\{T_k, T\})$. As before, we may assume that \hat{g} has bounded geometry of infinite order. By Theorem 2.2, we conclude that if $T_k < T$, then $|\text{Rm}(h_k(t))|_{h_k(t)}$ are bounded in $M \times [0, T_k)$. By Theorem 2.1, we see that one can extend $h_k(t)$ so that $T_k \geq T$. Given upper and lower bounds on $h_k(t)$ as above, we may conclude from Theorem 2.2, as in the proof of Theorem 4.1, that there is a smooth solution to the Kähler-Ricci flow $g(t)$ on $M \times (0, T)$ satisfying condition (a) and (c) from which we conclude, by Theorem 2.2 (i), that condition (b) is also satisfied. Also, by Theorem 4.1, we have that if g_0 is smooth and $\{h_{k,0}\}$ converges smoothly uniformly on compact sets, then $g(t)$ extends to a smooth solution on $M \times [0, T)$ such that $g(0) = g_0$. □

Corollary 4.1. *Let (M^n, \hat{g}) be a complete Kähler manifold with bounded curvature. Suppose u is real C^2 function on M such that $|\hat{\nabla}u| + |u| = o(r)$ and*

$$A^{-1}\hat{g} \leq \hat{g} + \sqrt{-1}\partial\bar{\partial}u \leq A\hat{g}$$

for some $A > 1$, where $\hat{\nabla}$ is the covariant derivative with respect to \hat{g} . Then the Kähler-Ricci flow with initial data $g_0 = \hat{g} + \sqrt{-1}\partial\bar{\partial}u$ has a solution $g(t)$ satisfying the conclusion of Theorem 4.2

(Here $f = o(r^k)$ represents a positive function on M such that $f(x)/d^k(x)$ approaches 0 as $d_p(x) \rightarrow \infty$ where $d(x)$ is the distance function from some fixed point in M relative to \hat{g}).

Proof. Let $r(x)$ be the distance with respect to \hat{g} from x to a fixed point $p \in M$. Since \hat{g} has bounded curvature, by [S2] (see also [T]), there is a smooth positive function ρ such that $\rho(x)$ is equivalent to $r(x)$ for $r(x) \geq 1$, $|\hat{\nabla}\rho|$ and $|\hat{\nabla}^2\rho|$ are bounded on M . Hence without loss of generality, in the following proof we may assume the distance function r , from some fixed $p \in M$ with respect to \hat{g} , is smooth with $|\hat{\nabla}r|, |\hat{\nabla}^2r|$ bounded on M . Let ϕ be a smooth function on \mathbb{R} such that $0 \leq \phi \leq 1$, $\phi(s) = 1$ for $s \leq 1$ and $\phi(s) = 0$ for $s \geq 2$, and $|\phi'| + |\phi''| \leq c_1$ for some c_1 . For any $k \geq 1$, let $\eta_k(x) = \phi(r(x)/k)$. Then $|\hat{\nabla}\eta_k|, |\hat{\nabla}^2\eta_k| \leq c_2/k$ on M for some constant c_2 independent of k . Now let $\{u_k\}$ be a sequence of smooth functions on M which converging to u , uniformly on compact subsets of M in the C^2 norm. Now for each k we have

$$(4.6) \quad \partial\bar{\partial}(\eta_k u_k) = \eta_k \partial\bar{\partial}u_k + u_k \partial\bar{\partial}\eta_k + \partial u_k \wedge \bar{\partial}\eta_k + \partial\eta_k \wedge \bar{\partial}u_k$$

Since $|\hat{\nabla}u| = o(r)$ and $u_k \rightarrow u$ uniformly on compact set in C^2 , we may assume that for each k we have $|\partial u_k \wedge \bar{\partial}\eta_k|_{\hat{g}}, |u_k \partial\bar{\partial}\eta_k|_{\hat{g}} = o_k(1)$ where $o_k(1)$ represents a positive function with upper bound on M approaching 0 as $k \rightarrow \infty$. Similarly, we may also assume that for each k we have

$$A\hat{g} \geq \sqrt{-1}\eta_k \partial\bar{\partial}u_k \geq (A^{-1} - 1)\hat{g},$$

and hence by (4.6) and the above discussion,

$$(A + o_k(1))\hat{g} \geq \sqrt{-1}\partial\bar{\partial}(\eta_k u_k) \geq (A^{-1} - 1 - o_k(1))\hat{g}.$$

Thus for sufficiently large k , $h_k = \hat{g} + \sqrt{-1}\partial\bar{\partial}(\eta_k u_k)$ is a Kähler metric such that

$$(2A)^{-1}\hat{g} \leq h_k \leq 2A\hat{g}.$$

In particular, for sufficiently large k h_k is complete, outside a compact set $h_k = \hat{g}$ and thus has bounded curvature, and h_k converges to g_0 uniformly on compact sets in C^0 . The corollary now follows from Theorem 4.2

□

Remark 1. Note that if $|\hat{\nabla}u| = o(1)$, then $|u| = o(r)$. This will imply that $|\hat{\nabla}u| + |u| = o(r)$. Hence if \hat{g} has bounded curvature and u is a

smooth function on M such that $\partial\bar{\partial}u$, u and $\hat{\nabla}u$ are bounded, then for some $\epsilon > 0$, the Kähler-Ricci flow with initial data $\hat{g} + \epsilon\sqrt{-1}\partial\bar{\partial}u$ has a short time solution.

From the proof of the theorem and Theorem 2.1

Corollary 4.2. *Let M^n be a complex noncompact manifold and let g_0 be a smooth complete Kähler metric with on M . Suppose there is a complete Kähler metric \hat{g} with bounded curvature and with holomorphic bisectional curvature bounded above by K such that $\hat{g} \leq g_0 \leq C\hat{g}$ for some $C \geq 1$. Let $T = 1/2nK$, where by convention we take $T = \infty$ when $K = 0$. Then the Kähler-Ricci flow has a solution $g(t)$ with initial data g_0 on $M \times [0, T)$ such that $g(t)$ has uniformly bounded curvature on $M \times [0, T']$ for all $T' < T$. Moreover,*

$$(4.7) \quad (1/n - 2Kt) \hat{g} \leq g(t).$$

on $M \times [0, T)$. Thus if $K = 0$ then the solution $g(t)$ exists for all time. In particular, the Kähler-Ricci flow with a complete Kähler metric with non-positive bounded curvature as initial data has long time solution.

Corollary 4.3. *Let M^n be a complex noncompact manifold and let h_0 be a bounded curvature Kähler metric on M such that $g_1 \leq h_0$ for some complete noncompact Kähler metric g_1 with bounded curvature such that the holomorphic bisectional curvature of g_1 is bounded above by K . Suppose in addition $h_0 \leq g_2$ where the holomorphic bisectional curvature of g_2 is bounded and non-positive. Let $T = 1/2nK$, where by convention we take $T = \infty$ when $K = 0$. Then the Kähler-Ricci flow has a solution $h(t)$ with initial data h_0 on $M \times [0, T)$ such that*

$$(4.8) \quad (1/n - 2Kt) g_1 \leq h(t).$$

on $M \times [0, T)$. Thus if $K = 0$ then the solution $h(t)$ exists for all time.

Proof. Consider the sequence of complete Kähler metrics

$$h_{k,0} = h_0 + \frac{1}{k}g_2.$$

Then for each k we have

$$\frac{1}{k}g_2 \leq h_{k,0} \leq 2g_2$$

Then $h_{k,0}$ converges uniformly on compact sets to h_0 . Moreover, for each k it is not hard to show that $h_{k,0}$ has bounded curvature. It follows from Theorem 4.2 and the fact that the holomorphic bisectional curvature of g_2 is non-positive, that for each k we have a longtime

solution $h_{k,0}(t)$ to (1.2) on $M \times [0, \infty)$ with initial condition $h_{k,0}$. On the other hand, this combined with the fact that for each k we have

$$g_1 \leq h_{k,0}$$

and Theorem 4.1 imply that we have a solution $h(t)$ to (1.2) on $M \times [0, T)$ with initial condition h_0 satisfying (4.8). \square

5. KÄHLER RICCI FLOW OF $U(N)$ INVARIANT METRICS ON \mathbb{C}^n

In this section we apply Theorems 4.2, 4.1 to $U(n)$ invariant metrics on \mathbb{C}^n .

5.1. Wu-Zheng's construction. We recall Wu-Zheng's construction in [WZ] of smooth $U(n)$ invariant metrics on \mathbb{C}^n . Begin with a smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$, and define functions $h, f : [0, \infty) \rightarrow \infty$ by

$$(5.1) \quad h(r) := Ce^{\int_0^r -\frac{\xi(t)}{t} dt}, \quad f(r) := \frac{1}{r} \int_0^r h(t) dt$$

where $h(0) = C > 0$ and $f(0) = h(0)$.

Now define a $U(n)$ invariant metric g on \mathbb{C}^n by

$$(5.2) \quad g_{i\bar{j}} = f(r)\delta_{ij} + f'(r)\bar{z}_i z_j.$$

where $g_{i\bar{j}}$ are the components of g in the standard coordinates $z = (z_1, \dots, z_n)$ on \mathbb{C}^n and $r = |z|^2$. Notice that a different choice of $h(0)$ simply corresponds to scaling the metric g above. In the following, we always take $C = 1$, i.e. $h(0) = 1$. Wu-Zheng [WZ] proved

Theorem 5.1. [Wu-Zheng]

1. The metric g above is complete iff

$$(5.3) \quad f > 0, \quad h > 0, \quad \int_0^\infty \frac{\sqrt{h}}{\sqrt{t}} dt = \infty$$

Conversely, up to scaling by a constant factor, every complete smooth $U(n)$ invariant Kähler metric on \mathbb{C}^n can be generated in this way.

2. At the point $z = (z_1, 0, \dots, 0)$, relative to the orthonormal frame $\{e_1 = \frac{1}{\sqrt{h}}\partial_{z_1}, e_2 = \frac{1}{\sqrt{f}}\partial_{z_2}, \dots, \frac{1}{\sqrt{f}}\partial_{z_n}\}$ with respect to $g_{i\bar{j}}$, we have

$$(i) \quad A = R_{1\bar{1}1\bar{1}} = \frac{\xi'}{h},$$

$$(ii) \quad B = R_{1\bar{1}i\bar{i}} = \frac{1}{(rf(r))^2} \int_0^r \xi'(t) \left(\int_0^t h(s) ds \right) dt,$$

(iii) $C = R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = \frac{2}{(rf(r))^2} \int_0^r h(t)\xi(t)dt$,
 where $2 \leq i \neq j \leq n$ and these are the only non-zero components of the curvature tensor at z except those obtained from A, B or C by the symmetric properties of R .

By the above construction, Wu-Zheng [WZ] proved the correspondence below for positively curved metrics, while Yang later showed in [Y] that this extends to a correspondence for non-negatively curved metrics.

Theorem 5.2. [Wu-Zheng, Yang] *There is a one to one correspondence between the set of all smooth complete $U(n)$ invariant Kähler metrics on \mathbb{C}^n with non-negative holomorphic bisectional curvature (modulo scaling by a constant factor) and the set of all smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ satisfying*

$$(5.4) \quad \xi(0) = 0, \quad \xi' \geq 0, \quad \xi \leq 1$$

Remark 2. One direction of the above correspondence is immediately obvious from Theorem 5.1 2 (i). In particular, it is obvious that if g has non-negative (non-positive) holomorphic bisectional curvature then $\xi' \geq 0$ ($\xi' \leq 0$).

5.2. Applications of Theorems 4.1 and 4.2 to $U(n)$ invariant metrics. We now apply Theorems 4.1 and 4.2 to $U(n)$ invariant metrics. First we have the following lemma.

Lemma 5.1. *Let $\xi : [0, \infty)$ be a smooth function with $\xi(0) = 0$. Suppose $\xi(r) = a$ for some constant $a \leq 1$ for all $r \geq r_0$. Then ξ generates a complete $U(n)$ invariant metric g such that the curvature of g approaches 0 as $x \rightarrow \infty$ on \mathbb{C}^n .*

Proof. For $r \geq r_0$,

$$\int_0^r \frac{\xi(t)}{t} dt = \int_0^{r_0} \frac{\xi(t)}{t} dt + a \log\left(\frac{r}{r_0}\right).$$

Hence

$$h(r) = c_1 r^{-a}$$

for some constant $c_1 > 0$ for all $r \geq r_0$. Since $a \leq 1$, it is easy to see that

$$\int_0^\infty \frac{\sqrt{h(r)}}{\sqrt{r}} dr = \infty.$$

Hence g is complete by Theorem 5.2.

Let A, B, C be the components of curvature of g as in Theorem 5.1. Then by the formulas in Theorem 5.1, we have $A(r) = 0$ for $r \geq r_0$. For $r \geq r_0$

$$\begin{aligned} |B| &\leq \frac{1}{(rf)^2} \int_0^r |\xi'(t)| \left(\int_0^t h(s) ds \right) dt \\ &\leq \frac{1}{(rf)^2} \left(\int_0^r h(t) dt \right) \int_0^r |\xi'(t)| dt \\ &\leq \frac{C}{rf} \end{aligned}$$

where $C = \int_0^{r_0} |\xi'_k(t)| dt$. Now

$$rf(r) = \int_0^r h(t) dt \geq c_2 + c_3 \log r$$

for some constants c_2, c_3 with $c_3 > 0$ because $a \leq 1$. Hence

$$\lim_{r \rightarrow \infty} B = 0.$$

Similarly, one can prove that $\lim_{r \rightarrow \infty} C = 0$. Hence the curvature of g tends to zero as $x \rightarrow \infty$. \square

Theorem 5.3. *Let $\hat{\xi} : [0, \infty) \rightarrow \mathbb{R}$, $\hat{\xi}(0) = 0$ generates a smooth complete bounded curvature $U(n)$ invariant Kähler metric \hat{g} . Suppose g_0 is a smooth complete $U(n)$ invariant Kähler metric on \mathbb{C}^n generated by a smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$ such that such that for all $r \geq 0$*

$$\int_0^r \frac{\xi - \hat{\xi}}{t} dt \leq C$$

for some $C > 0$ independent of r . Let K be the upper bound of the holomorphic bisectional curvature of \hat{g} . Then the Kähler-Ricci flow (1.2) has a smooth solution on $M \times [0, T)$ with initial condition $g(0) = g_0$, where $T = 1/(2nKe^C)$, where by convention $T = \infty$ if $K = 0$.

Proof. As ξ and $\hat{\xi}$ are smooth, for each $k \geq 0$ there exists a $\delta_k > 0$ and a smooth function “cutoff” function $\eta_k : (-\infty, \infty) \rightarrow \mathbb{R}$ satisfying

$$(5.5) \quad \eta_k(r) : \begin{cases} = 1 & \text{if } -\infty < r \leq k \\ 0 < \eta_k(r) < 1 & \text{if } k < r < k + \delta_k \\ = 0 & \text{if } k + \delta_k \leq r < \infty. \end{cases}$$

and

$$(5.6) \quad \int_k^{k+\delta_k} \left| \frac{(\xi - \hat{\xi})}{t} \right| dt \leq 1/k$$

for all k . Fix such a choice of η'_k s, and consider the sequence of functions $\{\xi_k\} : [0, \infty) \rightarrow \infty$ defined by

$$\xi_k(r) = \eta_k \xi + (1 - \eta_k) \hat{\xi}$$

and let h_k be the corresponding sequence of smooth $U(n)$ invariant Kähler metrics.

Consider the sequence of $U(n)$ invariant metrics h_k above.

$$\begin{aligned} \int_0^r \frac{\xi_k(t) - \hat{\xi}(t)}{t} dt &= \int_0^r \frac{\eta_k(\xi - \hat{\xi})}{t} dt \\ &= \begin{cases} \int_0^r \frac{\xi - \hat{\xi}}{t} dt, & \text{if } r \leq k; \\ \int_0^k \frac{\xi - \hat{\xi}}{t} dt + \alpha_k, & \text{if } r > k \end{cases} \end{aligned}$$

where

$$|\alpha_k| \leq \int_k^{k+\delta_k} \left| \frac{\xi - \hat{\xi}}{t} \right| dt \leq \frac{1}{k}.$$

where C is the constant in the hypothesis. Hence

$$\int_0^r \frac{\xi_k(t) - \hat{\xi}(t)}{t} dt \leq C + \frac{1}{k}.$$

This implies, by (5.1) and (5.2), that

$$\exp(-C - \frac{1}{k}) \hat{g} \leq h_k$$

In particular, h_k is complete. Also, from (5.1) and (5.2), we have

$$h_k \leq C_k \hat{g}$$

where $C_k = \exp\left(\int_0^{k+\delta_k} \left| \frac{\xi - \hat{\xi}}{t} \right| dt\right)$. It is also easy to see that h_k has bounded curvature for each k , and thus by Corollary 4.3, for each k there exists a solution $g_k(t)$ to (1.2) on $M \times [0, T_k)$ where $T_k = 1/(2n \exp(-C - \frac{1}{k}))$. Theorem 5.4 now follows from Theorem 4.1. \square

By Theorem 5.3, we have

Corollary 5.1. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric g_0 on \mathbb{C}^n generated by a smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$. If $\xi(r) \leq 1$ then the conclusion on Theorem 5.3. If in fact $\xi(r) \leq 0$ then the conclusion holds on Theorem 5.3 with $T = \infty$. In particular if g_0 has non-positive holomorphic bisectional curvature, thus $\xi'(r) \leq 0$ for all r , then the conclusion on Theorem 5.3 holds.*

Proof. By Lemma 5.1, $\hat{\xi}$ satisfying $\hat{\xi}(r) = c$ for some $c \leq 1$ and all r sufficiently large will generate a bounded curvature $U(n)$ invariant metric \hat{g} . Moreover, \hat{g} is flat if $\hat{\xi}(r) = 0$ identically. The result follows by choosing any $\hat{\xi}$ with $\hat{\xi}(r) = 1$ for sufficiently large r or choosing $\hat{\xi} = 0$ in case $\xi(r) \leq 0$ then applying Theorem 5.3. \square

We do not have any curvature bound on the solution in Theorem 5.3. In the next theorem, the solution also has some curvature bound.

Theorem 5.4. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric on \mathbb{C}^n generated by a smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$.*

Suppose ξ satisfies that

$$\int_a^r \frac{(c - \xi)}{t} dt, \quad \int_a^r \frac{(\xi - 1)}{t} dt \leq C$$

for some C and $c \leq 1$ independent of $0 < a \leq r$. Then there exists $T > 0$ such that the Kähler-Ricci flow (1.2) has a smooth solution on $M \times [0, T)$ with initial condition $g(0) = g_0$, where T depends only on c, C and n . Moreover, for every $l \geq 0$ there exists a constant C_l depending only on l, c, C and n such that

$$(5.7) \quad \sup_{p \in \mathbb{C}^n} \|\nabla^l \text{Rm}(p, t)\|_t^2 \leq \frac{C_l}{t^{l+2}}$$

on $\mathbb{C}^n \times (0, T)$.

If in addition,

$$\int_0^r \frac{\xi}{t} < C$$

for some constant C independent of r , then the above solution to Kähler-Ricci flow is defined on $M \times [0, \infty)$ and satisfies (5.7) on $\mathbb{C}^n \times (0, T)$ for some $T > 0$ depending only on c, C and n .

Corollary 5.2. *Let g_0 be a smooth complete $U(n)$ invariant Kähler metric g_0 on \mathbb{C}^n generated by a smooth function $\xi : [0, \infty) \rightarrow \mathbb{R}$ with $\xi(0) = 0$. If $c \leq \xi(r) \leq 1$ for some c then the conclusion on Theorem 5.4 holds. If in fact $c \leq \xi \leq 0$ for all r , then we have $T = \infty$ in the conclusion of Theorem 5.4.*

In particular, if $g_0(r)$ has non-negative holomorphic bisectional curvature, thus $\xi'(r) \geq 0$ for all r , then the conclusion on Theorem 5.4 holds, and $g(t)$ will have non-negative holomorphic bisectional curvature for all $t \in [0, T)$.

Proof. If $c \leq \xi(r) \leq 1$ (or $c \leq \xi(r) \leq 0$) for some c , then the conditions of part Theorem 5.4(b) clearly hold. In case $g_0(r)$ has non-negative

holomorphic bisectional curvature, the fact that $g(t)$ has non-negative bisectional curvature for all $t \in [0, T)$ provided g_0 was proved in [YZ]. \square

Remark 3. Let $\hat{\xi} : [0, \infty) \rightarrow \mathbb{R}$ to be smooth with $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1 + 1/\ln r$ for $r \geq 1$ say. Then from the proof of Proposition 5.1, it is not hard to see that the corresponding \hat{g} is complete with bounded curvature. Now it is easy to construct a smooth function $\xi \geq \hat{\xi}$ satisfying the assumptions in Theorem 5.3, where the corresponding g is complete with unbounded curvature. Thus ξ satisfies the assumptions in Theorem 5.3, while it is also easy to see that ξ does not satisfy the assumptions in Theorem 5.4.

5.3. Proof of Theorem 5.4. By Theorem 4.2, Theorem 5.4(b) will follow once we produce a sequence ξ_k and function $\hat{\xi}$ such that the corresponding $U(n)$ invariant Kähler metrics h_k , g_0 and \hat{g} satisfy the hypothesis of Theorem 4.2. We begin by proving the existence of such a function $\hat{\xi}$:

Proposition 5.1. *Under assumptions of Theorem 5.4, there exists $\hat{\xi}$ such that the corresponding $U(n)$ invariant metric \hat{g} has curvature bounded by a constant K_1*

$$(5.8) \quad K_2^{-1}\hat{g} \leq g_0 \leq K_2\hat{g}$$

on \mathbb{C}^n for some constant K_2 ; where K_1 depends only on c , and K_2 depends only on c and C which are the constants in the assumptions of Theorem 5.4.

We begin with a few Lemmas. Let $d = 1 - c > 0$.

Lemma 5.2. *Let $\epsilon > 0$ be such that $\ln \frac{3-\epsilon}{1+\epsilon} = 1$. For any $a > 0$, there exists a smooth function $\rho_a : [a, 3a] \rightarrow \mathbb{R}$ satisfying*

- (1) $\rho_a = 0$ near $t = a$; $\rho_a(3a) = d$ near $t = 3a$.
- (2) $0 \leq \rho'_a(r) \leq \frac{c_0 d}{r}$ for all $r \in (a, 3a)$

for some constant $c_0 > 0$ independent of a .

Proof. Note that $\epsilon = (3 - e)/(1 + e) < 0.15$. Since $\int_{a(1+\epsilon)}^{a(3-\epsilon)} \frac{d}{r} = d$, the Lemma follows by an appropriate smooth perturbation the continuous function

$$(5.9) \quad \tilde{\rho}_a = \begin{cases} 0 & \text{if } a \leq r < a(1 + \epsilon) \\ \int_{a(1+\epsilon)}^r \frac{d}{t} & \text{if } a(1 + \epsilon) \leq r < (3 - \epsilon)a \\ d & \text{if } (3 - \epsilon)a \leq r \leq 3a. \end{cases}$$

\square

Lemma 5.3. *Under assumptions of Theorem 5.4, if in addition,*

$$\int_1^r \frac{\xi - 1}{t} dt$$

is not bounded from below and

$$\int_1^r \frac{\xi - c}{t} dt$$

is not bounded from above, then for any $a > 0$, there there exists a least number $N_1(a) > 3a$, and a least number $N_2(a) > 3a$ such that

(i)

$$\int_a^{3a} \frac{(\xi - (1 - \rho_a))}{t} dt + \int_{3a}^{N_1(a)} \frac{(\xi - c)}{t} dt = (1 - c) \ln 3 + C + 1;$$

and

$$\int_a^{3a} \frac{(\xi - (1 - \rho_a))}{t} dt + \int_{3a}^r \frac{(\xi - c)}{t} dt \leq (1 - c) \ln 3 + C + 1;$$

for all $3a \leq r \leq N_1(a)$;

(ii)

$$\int_a^{3a} \frac{(\xi - (c + \rho_a))}{t} dt + \int_{3a}^{N_2(a)} \frac{(\xi - 1)}{t} dt = -(1 - c) \ln 3 - C - 1$$

$$\int_a^{3a} \frac{(\xi - (c + \rho_a))}{t} dt + \int_{3a}^r \frac{(\xi - 1)}{t} dt \geq -(1 - c) \ln 3 - C - 1$$

for all $3a \leq r \leq N_2(a)$.

where C is the constant in the assumption of Theorem 5.4(b) and ρ_a is given by the previous lemma.

Proof. We prove part (i). By the integral bounds in Theorem 5.4(b) and the fact that $c \leq 1 - \rho_a \leq 1$, we have

$$\int_a^{3a} \frac{(\xi - (1 - \rho_a))}{t} dt \leq (1 - c) \ln 3 + C.$$

Since $\int_1^r \frac{\xi - c}{t} dt$ is not bounded from above, $\int_a^r \frac{\xi - c}{t} dt$ is not bounded from above too. From this and the above inequality, we see that (i) is true. The proof of part (ii) is similar. \square

Proof of Proposition 5.1. Suppose $\int_1^r \frac{\xi - 1}{t} dt$ is bounded from below. Let $\hat{\xi}$ be such that $\hat{\xi}(0) = 0$ and $\hat{\xi}(r) = 1$ for $r \geq 1$. Then the $U(n)$ invariant Kähler metric \hat{g} will satisfy the conclusion of the lemma. Similarly, if $\int_1^r \frac{\xi - c}{t} dt$ is bounded from above, then $\hat{\xi}$ be such that $\hat{\xi}(0) = 0$ and

$\hat{\xi}(r) = c$ for $r \geq 1$ will generate a metric \hat{g} satisfying the conclusion of the lemma.

Suppose $\int_1^r \frac{\xi-1}{t} dt$ is not bounded from below and $\int_1^r \frac{\xi-c}{t} dt$ is not bounded from above, let $a_0 = 1$, $a_1 = N_1(a_0)$, $a_2 = N_2(a_1)$, $a_3 = N_1(a_2)$. In general, $a_{2k+1} = N_1(a_{2k})$, $a_{2k+2} = N_2(a_{2k+1})$, where N_1 and N_2 are defined in Lemma 5.3. Since $N_i(a) > 3a$, have

$$a_0 < a_1 < a_2 < \dots$$

and $a_k \rightarrow \infty$ as $k \rightarrow \infty$.

Now we define $\hat{\xi} : [0, \infty) \rightarrow \mathbb{R}$ as follows

- (i) Define $\hat{\xi}$ on $[0, 1] = [0, a_0]$ to be any smooth function with $\hat{\xi}(0) = 0$, $\hat{\xi}(r) = 1$ near $r = 1$.
- (ii) Define $\hat{\xi}$ on $[a_{2i}, a_{2i+1}]$ for any $i \geq 1$ by

$$(5.10) \quad \hat{\xi}(r) = \begin{cases} 1 - \rho_{a_{2i}} & \text{if } a_{2i} \leq r < 3a_{2i} \\ c & \text{if } 3a_{2i} \leq r \leq a_{2i+1} = N_1(a_{2i}). \end{cases}$$

- (iii) Define $\hat{\xi}$ on $[a_{2i-1}, a_{2i}]$ for any $i \geq 1$ by

$$(5.11) \quad \hat{\xi}(r) = \begin{cases} c + \rho_{a_{2i-1}} & \text{if } a_{2i-1} \leq r < 3a_{2i-1} \\ 1 & \text{if } 3a_{2i-1} \leq r \leq a_{2i} = N_2(a_{2i-1}). \end{cases}$$

By the definition of ρ_a , we see that $\hat{\xi}$ is smooth, $\hat{\xi}(0) = 0$ and $c \leq \hat{\xi} \leq 1$. Hence $\hat{\xi}$ generates a complete $U(n)$ invariant Kähler metric on \mathbb{C}^n .

By (5.1) and (5.2), proving (5.8) is equivalent to proving

$$(5.12) \quad \left| \int_0^r \frac{\xi - \hat{\xi}}{t} dt \right| \leq c_1$$

for some $c_1 > 0$ depending only on C, c and n appearing in the hypothesis of the Theorem and for all $r \geq 0$, which we now do. Let

$$c_2 = (1 - c) \ln 3 + C + 1$$

be the constant in Lemma 5.3. First note that

$$\int_{a_i}^{a_{i+2}} \frac{\xi - \hat{\xi}}{t} dt = 0$$

for all i by Lemma 5.3 and the definition of $\hat{\xi}$. For any $r \geq 1$, there is $i \geq 0$ such that $a_i \leq r < a_{i+1}$. Suppose $i = 2k$

$$\int_1^r \frac{\xi - \hat{\xi}}{t} dt = \int_{a_{2k}}^r \frac{\xi - \hat{\xi}}{t} dt \leq c_2$$

by the construction in Lemma 5.3 and (5.10). Suppose $i = 2k + 1$, then by (5.11)

$$\int_1^r \frac{\xi - \hat{\xi}}{t} dt = \int_{a_{2k+1}}^r \frac{\xi - \hat{\xi}}{t} dt \leq 2C + (1 - c) \ln 3$$

because $\xi - (c + \rho_{a_{2k+1}}) \leq \xi - 1 + (1 - c)$, and $\int_a^r \frac{\xi-1}{t} dt \leq C$ for all $0 < a < r$ by assumption. Hence

$$(5.13) \quad \int_1^r \frac{\xi - \hat{\xi}}{t} dt \leq 2C + (1 - c) \ln 3 + c_2.$$

Similarly one can prove that

$$(5.14) \quad \int_1^r \frac{\xi - \hat{\xi}}{t} dt \geq -2C - (1 - c) \ln 3 - c_2.$$

By (5.13) and (5.14) and the hypothesis of the Theorem, we conclude that (5.12) is true where c_1 depends only on C, c and n appearing in the hypothesis in the Theorem.

We now prove that the curvature components of \hat{g} are bounded using the formulas in Theorem 5.1.

$$(1) \quad A = \frac{\hat{\xi}'}{h}.$$

Since $\hat{h}(r) = e^{\int_0^r -\frac{\hat{\xi}}{t} dt}$, using the fact that $\hat{\xi} \leq 1$ we have $\hat{h}(r) \geq \frac{c_3}{r+1}$ for all $r \geq 0$ and some positive constant $c_3 > 0$ depending only on our choice of $\hat{\xi}$ on the interval $[0, 1]$. By the our construction of $\hat{\xi}$, and Lemma 5.2, we have $|\hat{\xi}'| \leq \frac{c_4}{r}$ for all r for some constant c_4 depending only on c in the hypothesis of the Theorem. Therefore, A is bounded by a constant depending only on c in the hypothesis of the Theorem.

$$(2)$$

$$\begin{aligned} B &= \frac{1}{(r\hat{f})^2} \int_0^r \hat{\xi}'(t) \left(\int_0^t \hat{h} \right) dt \\ &= \frac{1}{(r\hat{f})^2} \left(\hat{\xi}'(r) \int_0^r \hat{h}(t) dt - \int_0^r \hat{\xi}(t) \hat{h}(t) dt \right). \end{aligned}$$

Since $\hat{\xi}'$ is uniformly bounded depending only on c (by the our construction of $\hat{\xi}$, and Lemma 5.2), $c \leq \hat{\xi} \leq 1$, $h \geq c_3/(r+1)$ for $r \geq 0$, $r\hat{f}(r) = \int_0^r h \geq c_3 \ln(1+r)$, we conclude that B is bounded by a constant depending only on c in the hypothesis of the Theorem.

- (3) $C = \frac{1}{(rf)^2} \int_0^r \hat{\xi} \hat{h}$. By the above observations, we conclude that C is also likewise bounded by a constant depending only on c in the hypothesis of the Theorem.

□

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let \hat{g} be the $U(n)$ invariant Kähler metric with bounded curvature generated by $\hat{\xi}$ defined in Proposition 5.1, so that

$$(5.15) \quad K_2^{-1} \hat{g} \leq g_0 \leq K_2 \hat{g}$$

for some $K_2 > 0$ as in Proposition 5.1. As in the proof of Theorem 5.3, choose $\delta_k > 0$ and smooth function “cutoff” functions $\eta_k : (-\infty, \infty) \rightarrow \mathbb{R}$ satisfying

$$(5.16) \quad \eta_k(r) : \begin{cases} = 1 & \text{if } -\infty < r \leq k \\ 0 < \eta_k(r) < 1 & \text{if } k < r < k + \delta_k \\ = 0 & \text{if } k + \delta_k \leq r < \infty. \end{cases}$$

and

$$(5.17) \quad \int_k^{k+\delta_k} \left| \frac{(\xi - \hat{\xi})}{t} \right| dt \leq 1$$

for all k . Let $\{\xi_k\} : [0, \infty) \rightarrow \infty$ be defined by

$$\xi_k(r) = \eta_k \xi + (1 - \eta_k) \hat{\xi}.$$

Then as in the proof of Theorem 5.3, each ξ_k generates a $U(n)$ invariant Kähler metric h_k so that

$$(5.18) \quad K_3^{-1} \hat{g} \leq h_k \leq K_3 \hat{g}$$

for some constant $K_3 > 0$ depending only on c, C in the hypothesis of Theorem 5.4, for all k . Now recall that the curvature of \hat{g} is bounded by a constant K_1 as in Proposition 5.1, and thus by Theorem 2.1 we may assume without loss of generality that \hat{g} has bounded geometry of order infinity with bounds depending only on c and the order of covariant derivatives of curvature being taken. In particular, the formula of curvature in Theorem 5 implies that each h_k also has bounded curvature. We also clearly have $h_k \rightarrow g_0$ uniformly and smoothly on compact subsets of M . By Theorem 4.2, there is a solution $g(t)$ of the Kähler-Ricci flow with initial data g_0 on $M \times [0, T)$ for some $T > 0$ so that

$$\|\text{Rm}(g(t))\|_{g(t)}^2 \leq \frac{C''}{t}$$

for some $C'' > 0$ and for all $0 < t < T$ where T and C'' depend only on the constants c, C and n in the hypothesis of the Theorem. The estimates for $||\nabla^l \text{Rm}||$ for each $l \geq 0$ then follows from the general results of [S1].

If in addition that

$$\int_0^\infty \frac{\xi}{t} dt < C$$

then $g_0 \geq \alpha g_e$ for some constant $\alpha > 0$, where g_e is the Euclidean metric on \mathbb{C}^n . By (5.15) and (5.18),

$$g_k \geq \beta g_e$$

for some constant $\beta > 0$ for all k . Hence for each k the Kähler-Ricci flow with initial data g_k has a long time solution by Corollary 4.3 (a). It follows from Theorem 4.1 that there exists a longtime solution $g(t)$ to (1.2) with initial condition g_0 . Also, by (5.15) we conclude as above that $g(t)$ also satisfies (5.7) on $\mathbb{C}^n \times (0, T)$ for some $T > 0$ depending only on c, C and n . \square

The long time existence results in Theorem 5.4 are basically for $U(n)$ invariant metrics with non-positive curvature. The following Theorem gives a longtime existence result for $U(n)$ invariant metrics with non-negative curvature.

Theorem 5.5. *Suppose $\xi = a$ near infinity, with $a \leq 1$, then the Kähler-Ricci flow has long time solution $g(t)$ for the initial data g_0 generated by ξ . In general, if there is $C > 0$ such that*

$$(5.19) \quad -C \leq \int_1^r \frac{\xi - a}{t} dt \leq C$$

for some $a \leq 1$ for all $r > 1$ and such that $|\xi'| = o(r^{-a})$, then the Kähler-Ricci flow with initial data g_0 has long time solution $g(t)$ such that the curvature of $g(t)$ is uniformly bounded on $M \times [0, T]$ for all $T < \infty$.

Remark 4. If $a \leq 0$, then we have long time solution by Theorem 5.4. However, there is no curvature bound obtained for all t in that theorem. In that theorem, we can only conclude that the curvature of the solution is uniformly bounded in $M \times [0, T]$ for some $T > 0$.

Proof. Suppose (5.19) is true. We want to prove that the curvature of g tends to zero as $x \rightarrow \infty$. Consider the case that $a < 1$, then

$$h \geq c_1 r^{-a}$$

for large r for some $c_1 > 0$. Hence $rf \geq c_2 r^{1-a}$ for r large for some $c_2 > 0$. $|\xi'| = o(r^{-a})$ implies

$$\begin{aligned} |A| &= \left| \frac{\xi'}{h} \right| = o(1), \\ |B| &\leq \frac{1}{(rf)^2} \int_0^r h(t) dt \cdot \int_0^r |\xi'(t)| dt \leq \frac{1}{rf} \int_0^r |\xi'(t)| dt = o(1), \\ |C| &\leq \frac{2}{(rf)^2} \int_0^r h |\xi| = o(1) \end{aligned}$$

because $|\xi| = o(r^{1-a})$.

Suppose $a = 1$, then there is $c_3 > 0$ such that

$$h \geq \frac{c_3}{r}$$

for r large. So

$$rf \geq c_4 \log r$$

for some $c_4 > 0$ if r is large. We also have $|A|, |B|, |C| = o(1)$.

The Theorem now follows from the above curvature decay estimates, Lemma 5.4 below which implies the injectivity radius of g is bounded below on \mathbb{C}^n , and Theorem 2.3 because \mathbb{C}^n has a strictly pluri-subharmonic function. \square

Lemma 5.4. *Let $\xi(r) = a$ for all r sufficiently large and $a \leq 1$. Let g be the corresponding $U(n)$ invariant Kähler metric on \mathbb{C}^n . Then the injectivity radius of g is bounded below by a positive constant on \mathbb{C}^n*

Proof. We begin by assuming $a < 1$. Indeed, this will be sufficient for our applications. By the estimate in [CGT] and by the fact that the curvature of g is bounded by Lemma 5.1, in order to prove the injectivity radius of g is positive on \mathbb{C}^n it is sufficient to prove there is a constant $c > 0$ such that

$$V_g(B_g(x, 1)) \geq c$$

for all x where $B_g(x, 1)$ is the geodesic ball of radius 1 with center at x with respect to g . Let τ be the geodesic distance from the origin, then for $a < 1$ and $r = |z|^2 > r_0$.

$$(5.20) \quad \tau(z) = \int_0^r \frac{\sqrt{h}}{2\sqrt{s}} ds = c_1 + c_2 r^{\frac{1}{2}(1-a)}$$

for some constants c_1, c_2 with $c_2 > 0$. So

$$rf(r) = c_3 + c_4(\tau - c_1)^2$$

with $c_4 > 0$.

$$V(B_g(0; \tau)) = c_n (rf)^n = (c_3 + c_4(\tau - c_1)^2)^n.$$

where $\tau = \tau(r)$ is given by (5.20). Hence if τ is large, then

$$(5.21) \quad V_g(B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)) \geq c_5 \tau^{2n-1}$$

for some $c_5 > 0$ independent of τ . Let \mathcal{F} be a maximal disjoint family of $B_g(x, 1)$ with $x \in \partial B_g(0, \tau)$. Let $\mathcal{C} = \{x \mid B_g(x, 1) \in \mathcal{F} \text{ and let } N = N(\tau) = \#(\mathcal{C})$. We claim that $\bigcup_{x \in \mathcal{C}} B_g(x, 3) \supset B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)$. In fact, if $y \in B_g(0; \tau + 1) \setminus B_g(0; \tau - 1)$, then there is $y' \in \partial B_g(0; \tau)$ such that $d_g(y, y') < 1$. On the other hand, there is $x \in \mathcal{C}$ with $d_g(x, y') < 2$. From these the claim follows.

Since g is $U(n)$ invariant, $v = v(\tau) = V_g(B_g(x, 3))$ is constant for $x \in \partial B_g(0, \tau)$. Hence we have

$$Nv \geq c_5 \tau^{2n-1}$$

and

$$(5.22) \quad v \geq \frac{c_5}{N} \tau^{2n-1}.$$

By the expressions of h and f , on $\partial B_g(0, \tau) = \partial B_0(0, \sqrt{r})$, $c_6^{-1} r^{-a} g_0 \leq g \leq c_6 r^{-a} g_0$ for some $c_6 > 0$ if r is large, where $B_0(0, \sqrt{r})$ is the Euclidean ball with radius \sqrt{r} and center at the origin. Let $B_g^\tau(x, \rho)$ be the geodesic ball with respect to the intrinsic distance of $\partial B_g(0, \tau)$. Define $B_0^\tau(x, \rho)$ similarly with respect to g_0 .

Since $B_g(x, 1) \supset B_g^\tau(x, 1)$ and $B_0^\tau(x, c_6^{-1} r^{\frac{a}{2}}) \subset B_g^\tau(x, 1)$. Hence $\{B_0^\tau(x, c_6^{-1} r^{\frac{a}{2}}) \mid x \in \mathcal{C}\}$ is a disjoint family. Hence

$$NV_{g_0}(B_0^\tau(x, c_6^{-1} r^{\frac{a}{2}})) \leq V_{g_0}(\partial B_0(0, \sqrt{r})) = c_n r^{\frac{2n-1}{2}}.$$

where c_n is the volume of the unit sphere in \mathbb{C}^n . Let $\rho = r^{\frac{1}{2}}$, then the volume of the geodesic ball of radius s_0 in $\partial B_0(0, \rho)$ is

$$c_n \rho^{n-2} \int_0^{s_0} \sin^{2n-2} \frac{s}{\rho} ds.$$

where c_n is a positive constant depending on n . Let $s_0 = c_6^{-1} r^{\frac{a}{2}}$. Then $s_0/\rho \rightarrow 0$ as $r \rightarrow \infty$. Hence for r large,

$$\begin{aligned} V_{g_0}(B_0^\tau(x, c_6^{-1} r^{\frac{a}{2}})) &\geq c_7 \int_0^{s_0} s^{2n-2} ds \\ &= c_8 s_0^{2n-1}. \end{aligned}$$

Hence

$$\begin{aligned} v &\geq c_5 N^{-1} \tau^{2n-1} \\ &\geq c_n^{-1} c_5 c_8 \tau^{2n-1} r^{-\frac{2n-1}{2}} s_0^{2n-1} \\ &\geq c_9 \end{aligned}$$

for some positive constant c_9 independent of τ .

We now consider the case when $a = 1$. Consider Cao's cigar soliton \tilde{g} which is a complete $U(n)$ invariant Kähler metric on \mathbb{C}^n . It is shown in [WZ] \tilde{g} has positive sectional curvatures and is generated by $\tilde{\xi}$ satisfying

$$(5.23) \quad \int_0^\infty \frac{\xi - \tilde{\xi}}{t} dt < \infty$$

since $\xi(r) = 1$ for sufficiently large r (see Theorem 3 in [WZ]).

In particular, by (5.1) and (5.2) it follows that g and \tilde{g} are uniformly equivalent and thus

$$V_g(B_g(p, 1)) \geq CV_{\tilde{g}}(B_{\tilde{g}}(p, 1))$$

for some $C > 0$ for all $p \in \mathbb{C}^n$ and for some constant C independent of p . To bound the injectivity radius of g from below, it suffices to prove that the volume in the RHS above is uniformly bounded below. This follows from [GM] since \tilde{g} is complete, and has bounded positive sectional curvatures. For completeness, we include a proof below that \tilde{g} has bounded curvature. By Wu-Zheng:

Let $\tilde{\phi} = r\tilde{f}$ and $t = \log r$. $\tilde{\phi}' = r\tilde{h}$. Hence $\tilde{\phi} > 0$, $\tilde{\phi}' > 0$ for $t > -\infty$. Here all primes on $\tilde{\phi}$ are with respect to t . Since $A, B, C > 0$, we only need to prove that A, B, C are bounded from above. It is sufficient to prove that A, B, C are bounded from above for $t \geq 0$. For $t \geq 0$, by [WZ, §4]

$$A = n(1 + \frac{n-1}{\tilde{\phi}}) - \tilde{\phi}' \left(1 + \frac{2(n-1)}{\tilde{\phi}} + \frac{n(n-1)}{\tilde{\phi}^2} \right) \leq n(1 + \frac{n-1}{\tilde{\phi}(0)}),$$

because $\tilde{\phi}' > 0$, $\tilde{\phi} > 0$. So A is bounded.

$$B = \frac{1}{(r\tilde{f})^2} \int_0^r \frac{d\tilde{\xi}}{dr} \left(\int_0^t \tilde{h}(s) ds \right) dt \leq \frac{1}{r\tilde{f}}$$

because $\frac{d\tilde{\xi}}{dr} > 0$, $\tilde{\xi}(r) \leq 1$. On the other hand by (5.23), $\tilde{h}(r) \geq cr^{-1}$ for $r \geq 1$. Hence $r\tilde{f} \sim c \log r$. So B is bounded. Similarly, C is also bounded.

□

Remark 5. In case $1 > a \geq 0$, we may simply compare g with a metric \tilde{g} with nonnegative bisectional curvature generated by $\tilde{\xi}$ with $\tilde{\xi} = a$ near infinity. In this case, \tilde{g} has maximum volume growth by [WZ]. Hence each geodesic ball of radius 1 is bounded below by a constant which is uniform for all points. So this is also true for g .

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